Theoretical Question 1: The Shockley-James Paradox

SOLUTION

a. The magnetic field created by the large loop at its center is:

$$B = \frac{\mu_0 I_2}{2R}$$

Since $r \ll R$, this is the field throughout the area of the small loop. Therefore, the flux through the small loop is given by:

$$\Phi_{B1} = \pi r^2 B = \frac{\pi \mu_0 r^2 I_2}{2R}$$

The mutual inductance is then given by:

$$M_{21} = \frac{\pi \mu_0 r^2}{2R}$$

b. Since $M_{12} = M_{21} = M$, we have:

$$\Phi_{B2} = M I_1 = \frac{\pi \mu_0 r^2 I_1}{2R}$$

Taking the derivative with respect to time, this becomes:

$$\varepsilon_2 = \frac{\pi \mu_0 r^2 \dot{I}_1}{2R}$$

c. The EMF is work per unit charge, while the electric field is force per unit charge. Therefore:

$$E = \frac{\varepsilon_2}{2\pi R} = \frac{\mu_0 r^2 \dot{I}_1}{4R^2}$$

d. The electric field from part (c) leads to a force:

$$F = EQ = \frac{\mu_0 r^2 Q \dot{I}}{4R^2}$$

Integrating over dt (and disregarding the sign), we get the impulse:

$$\Delta p = \frac{\mu_0 r^2 I Q}{4R^2}$$

e. The current can be written as:

$$I = nAqv$$

where v is the charge carriers' velocity. We therefore have:

$$v = \frac{I}{nAq}$$

The momentum is then given by:

$$p = \gamma mnAlv = \frac{mnAlv}{\sqrt{1 - v^2/c^2}} = \frac{mIl}{q}\left(1 - \left(\frac{I}{nAqc}\right)^2\right)^{-1/2}$$

where γ is the Lorentz factor associated with v.

f. The hidden momentum is due to the charge carriers in the two vertical sides of the loop. Let m be the mass of the charge carriers, let q be their charge, and let $\Delta U = kQql/R^2$ be the potential energy difference for a charge carrier between the two sides. Denote the longitudinal densities and velocities of the charges in the two sides by λ_1, v_1, λ_2 and v_2. Let γ_1 and γ_2 be the appropriate Lorentz factors. From the constant value of the current, we have:

$$q\lambda_1 v_1 = q\lambda_2 v_2 = I$$

Energy conservation for the charge carriers passing from one side to the other reads:

$$(\gamma_2 - \gamma_1) \cdot mc^2 = \Delta U$$

The total momentum now reads:

$$p_{hid} = p_2 - p_1 = ml(\gamma_2\lambda_2 v_2 - \gamma_1\lambda_1 v_1) = \frac{mIl}{q}(\gamma_2 - \gamma_1) = \frac{Il\Delta U}{qc^2} = \frac{kQIl^2}{R^2c^2}$$

Note that all the microscopic quantities m, q, λ_i and v_i have dropped out.

g) In part (d), the magnetic moment is $\mu = \pi r^2 I$, and we get:

$$\Delta p = \frac{\mu_0 Q\mu}{4\pi R^2}$$

In part (f), the magnetic moment is $\mu = l^2 I$, and we get:

$$p_{hid} = \frac{kQ\mu}{R^2c^2} = \frac{\mu_0 Q\mu}{4\pi R^2}$$

We see that the results are identical.

h) The answer is (A)+(C). (A) is true because ΔU between the near side and the far side of the loop vanishes. (B) cannot be true, because the back-reaction of the induced charges on the external charge is a higher-order effect; for instance, it involves higher powers of Q. Then the conservation of center-of-mass velocity requires that (C) is true.