Solution and Marking Scheme

Theory

I. Satellite's orbit transfer

a) $$\frac{mu_0^{\ 2}}{R_0} = \frac{GMm}{R_0^{\ 2}}, \qquad u_0 = \sqrt{\frac{GM}{R_0}}$$ (1 point)

b) conservation of angular momentum: $mu_1 R_0 = mu_2 R_1$

conservation of energy: $\frac{1}{2} mu_2^{\ 2} - \frac{GMm}{R_1} = \frac{1}{2} mu_1^{\ 2} - \frac{GMm}{R_0}$

$$\left[\left(\frac{R_0}{R_1}\right)^2 - 1\right] u_1^{\ 2} = 2GM\left[\frac{1}{R_1} - \frac{1}{R_0}\right]$$

$$\frac{(R_0 - R_1)(R_0 + R_1)}{R_1^{\ 2}} u_1^{\ 2} = (2GM)\frac{(R_0 - R_1)}{R_0 R_1}$$

$$u_1 = \sqrt{\frac{GM}{R_0}}\sqrt{\frac{2R_1}{R_1 + R_0}} = u_0\sqrt{\frac{2R_1}{R_1 + R_0}}$$ (2 points)

c) $$\lim_{R_1 \to \infty} u_1 = \sqrt{2}\, u_0$$ (1 point)

d) $$u_2 = u_1 \frac{R_0}{R_1} = u_0 \frac{\sqrt{2}\, R_0}{\sqrt{R_1 (R_1 + R_0)}}$$ (1 point)

e) $$u_3 = \sqrt{\frac{GM}{R_1}} = \sqrt{\frac{GM}{R_0}}\sqrt{\frac{R_0}{R_1}} = u_0\sqrt{\frac{R_0}{R_1}}$$

$$= \sqrt{\frac{R_0}{R_1}}\sqrt{\frac{R_1(R_1 + R_0)}{\sqrt{2} R_0}}\, u_2$$

$$u_3 = u_2\sqrt{\frac{R_1 + R_0}{2R_0}}$$ (1 point)

f) (3 points) combining equations (1) and (2) :

$$\frac{d^2}{dt^2} r - \frac{C/m}{r^3} = -\frac{GM}{r^2}$$

and for the circular orbit of radius R_1 we have $\frac{C}{m} = GMR_1$

hence $$\frac{d^2}{dt^2} r - \frac{GMR_1}{r^3} = -\frac{GM}{r^2}$$

putting $r = R_1 + \eta$, where $\eta << R_1$

$$\therefore \frac{d^2}{dt^2}\eta - \frac{GMR_1}{R_1^{\ 3}\left(1 + \frac{\eta}{R_1}\right)^3} = -\frac{GM}{R_1^{\ 2}\left(1 + \frac{\eta}{R_1}\right)^2}$$

$$\frac{d^2}{dt^2}\eta - \frac{GM}{R_1^{\ 2}}\left(1 - 3\frac{\eta}{R_1}\right) \approx -\frac{GM}{R_1^{\ 2}}\left(1 - 2\frac{\eta}{R_1}\right)$$

$$\frac{d^2}{dt^2}\eta \approx -\frac{GM}{R_1^{\ 3}}\eta$$

the frequency of oscillation about mean distance is $f = \frac{1}{2\pi}\sqrt{\frac{GM}{R_1^{\ 3}}}$

the period $T = \frac{1}{f} = 2\pi\sqrt{\frac{R_1^{\ 3}}{GM}}$

Note that this period is the same as the orbital period

h) (1 point)